Level statistics of a pseudo-Hermitian Dicke model

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A non-Hermitian operator that is related to its adjoint through a similarity transformation is defined as a pseudo-Hermitian operator. We study the level statistics of a pseudo-Hermitian Dicke Hamiltonian that undergoes quantum phase transition (QPT). We find that the level-spacing distribution of this Hamiltonian near the integrable limit is close to Poisson distribution, while it is Wigner distribution for the ranges of the parameters for which the Hamiltonian is nonintegrable. We show that the assertion in the context of the standard Dicke model that QPT is a precursor to a change in the level statistics is not valid in general.

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The study on statistics of energy levels in quantum many-body systems has a long history [1,2]. The statistical analysis based on the random matrix theory (RMT) has been applied to characterize quantum chaos and to investigate the integrability of a quantum system. In particular, it has been conjectured [3] that the level-spacing distribution of an integrable Hermitian Hamiltonian should be described by the Poisson distribution: $P_{\mathrm{P}}(s)=\exp(-s)$. On the other hand, if the system is nonintegrable, the level-spacing distribution of the Hermitian Hamiltonian should be given by the Wigner distribution, i.e., the Wigner surmise for the Gaussian orthogonal ensemble (GOE): $P_{\mathrm{W}}(s)=(\pi s/2)\exp(-\pi s^2/4)$. Although there is no rigorous proof of the Bohigas-Giannoni-Schimdt (BGS) conjecture [3] for quantum systems, it has been numerically confirmed for a variety of many-body Hamiltonian [2] and also theoretically in the semiclassical limit [4].

The RMT without the constraint of Hermiticity was introduced by Ginibre [5] and, currently, is an active field of research [6]. The non-Hermitian RMT exhibits generic statistical behavior of quantized dissipative systems. The integrable case corresponds to the Poisson process on the plane, while a cubic repulsion is a signature of quantum chaotic scattering [6]. An interesting result due to Ginibre [5] is that the probability density function for the eigenvalues of real Gaussian random nonsymmetric matrices with all the eigenvalues being real is identical to the GOE and, consequently, the level-spacing distribution is given by $P_{\mathrm{W}}(s)$. If all the eigenvalues of a real nonsymmetric matrix M are real, it can be shown that the same matrix can be mapped to its transpose through a similarity transformation. In particular, $M^T=(XX^T)^{-1}M(XX^T)$, where the real matrix X diagonalizes M with entirely real eigenvalues E, i.e., $M=XEX^{-1}$. This shows that Ginibre's ensemble of real nonsymmetric matrices belong to the class of operators known as pseudo-Hermitian operator, i.e., an operator that is related to its adjoint through a similarity transformation.

The study on pseudo-Hermitian operators has received considerable attention recently in connection with the pioneering work of Bender and Boettcher [7], showing that non-Hermitian operators with unbroken $\mathcal{PT}$ symmetry admit entirely real spectra. This has opened up several new directions [8–12] in the study of pseudo-Hermitian operators. One of the significant developments is the construction of pseudo-Hermitian RMT with pseudounitary symmetry [13]. The degree of level repulsion is different from that of previously known Gaussian orthogonal, unitary, and symplectic ensembles and it seems to point out a new universality class. Moreover, unlike the Ginibre ensembles (except for the exceptional case discussed above), non-Hermitian RMT with pseudounitary symmetry describes nondissipative systems.

In spite of all the above developments, a criterion to characterize quantum chaos and to investigate the integrability of a pseudo-Hermitian operator using the level statistics based on RMT is still lacking. Pseudo-Hermitian operators with entirely real spectra can be shown to be Hermitian with respect to some modified inner product in the Hilbert space [8]. The effect of the modified inner product in the Hilbert space is to have a modified symplectic structure for the corresponding classical system. It may be noted here that a fixed modified inner product in the Hilbert space or the corresponding symplectic structure is not universal for pseudo-Hermitian systems. It varies from one system to another and, in general, it is a difficult problem to identify the proper inner product for a given pseudo-Hermitian system admitting an entirely real spectrum. It is thus not clear *a priori* whether the standard semiclassical analysis in support of the BGS conjecture for quantum systems with a standard inner product in the Hilbert space will remain unchanged for generic pseudo-Hermitian systems or not. In the absence of any theoretical support for the validity of the BGS conjecture for pseudo-Hermitian systems, it may be worth looking for numerical evidences.

The purpose of this paper is to present numerical evidence to show that the level-spacing distribution of a nonintegrable pseudo-Hermitian Dicke Hamiltonian (DH) with an entirely real spectrum is described by the Wigner distribution. On the other hand, it approaches to the Poisson distribution for the

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parameters of the model close to the integrable limit.

We consider a pseudo-Hermitian DH that has been shown recently to undergo QPT [12],

$$H = \omega a^\dagger a + \omega_0 J_z + \frac{\alpha}{\sqrt{2j}} e^{i\xi_1} J_- a^\dagger + \frac{\beta}{\sqrt{2j}} e^{-i\xi_1} J_+ a + \frac{\gamma}{\sqrt{2j}} e^{i\xi_2} J_- a + \frac{\delta}{\sqrt{2j}} e^{-i\xi_2} J_+ a^\dagger, \tag{1}$$

where ω, ω_0, α, β, γ, δ, ξ_1, and ξ_2 are real parameters and j is the total spin-angular momentum. The operators a, $a^\dagger$ are the standard bosonic annihilation-creation operators and J_z, $J_\pm$ are the generators of the SU(2) algebra,

$$[a,a^\dagger] = 1,$$

$$[J_+,J_-] = 2J_z, \quad [J_z,J_\pm] = \pm J_\pm. \tag{2}$$

The Hamiltonian H commutes with the parity operator Π,

$$\Pi = e^{i\pi\hat{N}}, \quad \hat{N} = a^\dagger a + J_z + j. \tag{3}$$

The eigenstates of H have definite parity depending on whether the eigenvalues of the operator $\hat{N}$ are odd or even. The Hamiltonian (1) reduces to the standard DH for $\xi_1=\xi_2=0$ and $\alpha=\beta=\gamma=\delta$. The DH has been studied extensively from the viewpoint of QPT [14–16], level-statistics [16], quantum entanglement [17,18], exact solvability [19], and two-dimensional semiconductor physics [20].

The non-Hermitian Hamiltonian H can be mapped to a Hermitian Hamiltonian $\mathcal{H}=\rho H\rho^{-1}$ through a similarity transformation when the following relation is satisfied [12],

$$\alpha\delta - \beta\gamma = 0. \tag{4}$$

The operator ρ and $\mathcal{H}$ have the following forms [12]:

$$\rho = \exp\left[\frac{1}{4}\ln\left(\frac{\alpha\gamma}{\beta\delta}\right)(J_z + j)\right], \quad \frac{\alpha}{\beta} > 0, \quad \frac{\gamma}{\delta} > 0,$$

$$\mathcal{H} = \omega a^\dagger a + \omega_0 J_z + \sqrt{\frac{\alpha\beta}{2j}}(e^{i\xi_1} J_- a^\dagger + e^{-i\xi_1} J_+ a) + \sqrt{\frac{\gamma\delta}{2j}}(e^{i\xi_2} J_- a + e^{-i\xi_2} J_+ a^\dagger). \tag{5}$$

The Hamiltonian H that is non-Hermitian under the Dirac-Hermiticity condition becomes Hermitian with respect to the modified inner product defined in the Hilbert space as, $\langle\langle u,v\rangle\rangle_{\eta_+} := \langle u,\eta_+ v\rangle$, where the metric $\eta_+ := \rho^2$. In particular,

$$\langle u|Hv\rangle \neq \langle Hu|v\rangle, \quad \langle\langle u|Hv\rangle\rangle_{\eta_+} = \langle\langle Hu|v\rangle\rangle_{\eta_+}. \tag{6}$$

Thus, with the modified inner product, the results of a Hermitian Hamiltonian follow automatically. Note that $\langle u|\mathcal{H}v\rangle = \langle\mathcal{H}u|v\rangle$. We refer to Ref. [12] for further details.

In this system, when j is finite, the parity Π is a good quantum number. Two states with different parity do not interact with each other. In other words, we can concentrate on the states with either positive or negative Π. Here, we consider the positive-parity states.

The level-spacing distributions are given by the probability function $P(s)$ of nearest-neighbor spacings $s_i=x_{i+1}-x_i$, where x_i are unfolded eigenvalues. In order to characterize the level-spacing distribution, we employ the quantity,

$$\eta \equiv \left|\frac{\int_0^{s_0}[P(s) - P_{\mathrm{W}}(s)]ds}{\int_0^{s_0}[P_{\mathrm{P}}(s) - P_{\mathrm{W}}(s)]ds}\right|, \tag{7}$$

where $s_0=0.472\,9\ldots$ is the intersection point of $P_{\mathrm{P}}(s)$ and $P_{\mathrm{W}}(s)$. We have $\eta=1$ when $P(s)=P_{\mathrm{P}}(s)$, and $\eta=0$ when $P(s)=P_{\mathrm{W}}(s)$.

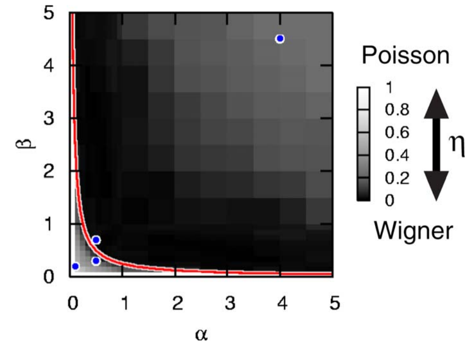

FIG. 1. (Color online) Phase diagram of η, which is defined by Eq. (7), in a special case where $\alpha=\gamma$ and $\beta=\delta$. The solid curve corresponds to the critical line $\alpha\beta=1/4$. Solid circles correspond to the level-spacing distributions in Fig. 2.

In the following, we set $\omega=\omega_0=1$ and $\xi_1=\xi_2=0$ for convenience. A further choice of $\gamma=\frac{\alpha}{n}$ essentially fixes δ as $\delta=\frac{\beta}{n}$ due to the pseudo-Hermiticity condition (4), where $n(\neq -1)$ is a real number. With this parametrization, the Hamiltonian H can be rewritten as

$$H = a^\dagger a + J_z + \frac{1}{\sqrt{2j}}\left(\alpha J_- a^\dagger + \beta J_+ a + \frac{\alpha}{n} J_- a + \frac{\beta}{n} J_+ a^\dagger\right). \tag{8}$$

The equivalent Hermitian Hamiltonian $\mathcal{H}$ has the following form:

$$\mathcal{H} = a^\dagger a + J_z + \sqrt{\frac{\alpha\beta}{2j}}\left[J_- a^\dagger + J_+ a + \frac{1}{n}(J_- a + J_+ a^\dagger)\right]. \tag{9}$$

The total spin-angular momentum j should be large enough to obtain proper results of level statistics. If j is very small ($j\sim 1$) because of a kind of finite-size effects, level statistics shows no universal ensembles [16]. In our numerical calculation, $j=10$ unless specifically mentioned.

Figure 1 exhibits the phase diagram of η for H in Eq. (8) with $n=1$. At the critical line $\alpha\beta=1/4$, η rapidly changes. For $\alpha\beta < 1/4$, level statistics is almost Poissonian as seen in Fig. 2(a) [21]. As $\alpha\beta$ increases, $P(s)$ changes from the Poisson to Wigner distributions. It gives an intermediate distribution, e.g., Fig. 2(b). For $\alpha\beta > 1/4$, level-spacing distribution is almost given by the Wigner distribution, as shown in Fig. 2(c). However, as $\alpha\beta$ increases further, η gradually increases. In other words, level statistics gradually changes from the Wigner distribution to Poissonian one again as $\alpha\beta$ becomes very large. Figure 2(d) is an example of an intermediate distribution for large $\alpha\beta$. The behavior of η along the line $\alpha=\beta$ in Fig. 1 corresponds to that of Ref. [16].

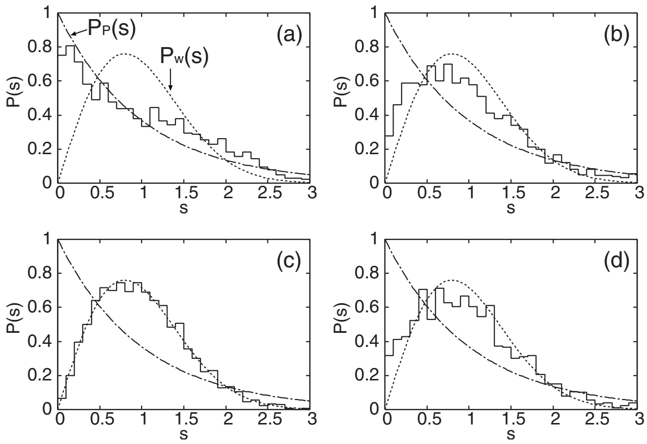

FIG. 2. Level-spacing distributions for the Hamiltonian (8) with $n=1$. (a) Almost Poisson distribution at $\alpha=\gamma=0.1$ and $\beta=\delta=0.2$, (b) intermediate distribution at $\alpha=\gamma=0.5$ and $\beta=\delta=0.3$, (c) Wigner distribution at $\alpha=\gamma=0.5$ and $\beta=\delta=0.7$, and (d) deformed Wigner distribution at $\alpha=\gamma=4$ and $\beta=\delta=4.5$.

The Hamiltonian $\mathcal{H}$ in Eq. (9) corresponds to the standard DH for $n=1$ and has been studied in some detail in Ref. [16]. Based on the results of [16] and the numerical findings for H as described above, we suggest the following:

(i) The criteria to distinguish between integrable and nonintegrable phases of a Hermitian Hamiltonian are valid also for a pseudo-Hermitian Hamiltonian. The quantum Hamiltonian $\mathcal{H}$ and H have the identical eigenvalues since they are related to each other through a similarity transformation. Thus, both H and $\mathcal{H}$ show similar changes in the level-spacing distributions as a function of $\alpha\beta$.

(ii) The onset of quantum chaos in a pseudo-Hermitian Hamiltonian is manifested by a change in the level statistics from Poissonian to Wigner distribution. It is known that the semiclassical Hermitian Hamiltonian corresponding to $\mathcal{H}$ shows chaotic behavior for $\alpha\beta>\frac{1}{4}$ and regular periodic orbits are obtained for $\alpha\beta<\frac{1}{4}$ [16]. It is also the case for the Hamiltonian H in the semiclassical limit. In fact, a non-Hermitian Hamiltonian and its equivalent Hermitian Hamiltonian describe the same physics in the classical limit within the formalism of pseudo-Hermitian quantum physics and the correspondence principle [8].

We now present numerical results for other values of n. Figures 3(a) and 3(b) are phase diagrams of η for $n=\frac{1}{2}$ and $n=2$, respectively. Changes in level statistics appear around the critical line determined by $\alpha\beta=\frac{n^2}{(n+1)^2}$ for both the cases, i.e., the physical picture is identical to the case of $n=1$. However, the change of η around the critical line is small in Figs. 3(c) and 3(d), which exhibit phase diagrams of η for $n=\frac{1}{3}$ and $n=3$, respectively. In fact, level statistics does not show clear Wigner behavior for $\alpha\beta>\frac{n^2}{(n+1)^2}$ with $n=\frac{1}{3}$ and $n=3$.

In order to see the change of η clearly, we depict η as a function of n for $\alpha\beta=1$ in Fig. 3(e), where the curve of η for $j=10$ is compared with that for $j=30$. The nonzero η behavior for $n>2$ and $n<\frac{1}{2}$ in Fig. 3(e) indicates that Poisson behavior continues beyond the critical line in the phase diagram of η for those regimes of n. The behavior is a manifestation of the fact that the system becomes close to the integrable limit, if $|n|\ll 1$ or $|n|\gg 1$. One plausible explanation to understand the origin of the precise critical values of n (i.e., $\frac{1}{2}$ and 2) may lie in the nonapplicability of perturbation techniques by treating either the counter-rotating (for $1<n\leq 2$) or the rotating terms (for $1>n\geq\frac{1}{2}$) as perturbation.

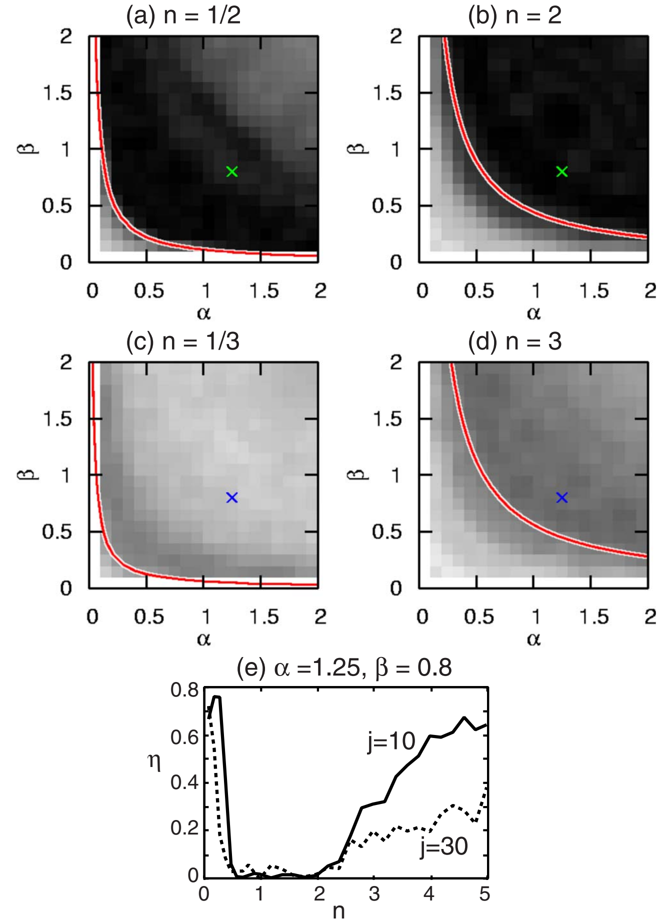

FIG. 3. (Color online) Phase diagram of η, which is defined by Eq. (7) for $\gamma=\alpha/n$ and $\delta=\beta/n$; (a) $n=1/2$, (b) $n=2$, (c) $n=1/3$, and (d) $n=3$. The solid curve in each panel is given by $\alpha\beta=n^2/(n+1)^2$. (e) Dependence of η plotted against n for $\alpha=1.25$ and $\beta=0.8$. The parameter combination is shown in (a)–(d) by the points ($\times$).

A comment is in order at this point. The pseudo-Hermitian Dicke model is known to undergo QPT with the critical line determined by the equation $\alpha\beta=\frac{n^2}{(n+1)^2}$ [12]. We suggest that the value of η, which characterizes the level statistics, is a possible measure to estimate the onset of the QPT not only for the Hermitian Hamiltonian $\mathcal{H}$, but, also for the quasi-Hermitian Hamiltonian H. However, the usual assertion [16] within the context of the standard Dicke model that QPT is a precursor to a change in the level statistics is not valid in general for the pseudo-Hermitian Dicke model, which has a larger parameter space. According to the relevant assertion in Ref. [16], level statistics is given by the Wigner distribution for $\alpha\beta>\frac{n^2}{(n+1)^2}$. For any positive n, the inequality is satisfied when $\alpha\beta=1$. Therefore, if the assertion was always valid, η would be always zero (or very small) for $n>0$, which is certainly not the case for $n>2$ and $n<\frac{1}{2}$ in

Fig. 3(e). The assertion is valid for the pseudo-Hermitian Dicke model for $\frac{1}{2}<n<2$ of which the standard Dicke model corresponding to $n=1$ appears as a special case.

We conclude with the following:

(i) Based on our numerical results, we conjecture that the level-spacing distribution comes close to the Poisson distribution $P_{\mathrm{P}}(s)$ as the system approaches the integrable limit, while for the nonintegrable pseudo-Hermitian Hamiltonian it should be described by the Wigner distribution $P_{\mathrm{W}}(s)$.

(ii) We have also shown that the assertion [16] that QPT is a precursor to a change in the level statistics in the standard Dicke model is not valid in general for the pseudo-Hermitian Dicke model, which has a larger parameter space. The assertion holds true for the pseudo-Hermitian Dicke model only for a limited range of the parameter space.

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[1] M. L. Mehta, *Random Matrices and the Statistical Theory of Energy Levels* (Academic, New York, 1967).
[2] F. Haake, *Quantum Signatures of Chaos* (Springer-Verlag, Berlin, 1992).
[3] O. Bohigas, M. J. Giannoni, and C. Schmit, Phys. Rev. Lett. **52**, 1 (1984).
[4] S. Heusler, S. Müller, A. Altland, P. Braun, and F. Haake, Phys. Rev. Lett. **98**, 044103 (2007).
[5] J. Ginibre, J. Math. Phys. **6**, 440 (1965).
[6] R. Grobe, F. Haake, and H.-J. Sommers, Phys. Rev. Lett. **61**, 1899 (1988); W. John, B. Milek, H. Schanz, and P. Seba, *ibid.* **67**, 1949 (1991); G. Akemann and E. Kanzieper, J. Stat. Phys. **129**, 1159 (2007), and references therein.
[7] C. M. Bender, Contemp. Phys. **46**, 277 (2005); C. M. Bender, D. C. Brody, and H. F. Jones, Am. J. Phys. **71**, 1095 (2003); C. M. Bender and S. Boettcher, Phys. Rev. Lett. **80**, 5243 (1998); J. Phys. A **31**, L273 (1998); C. M. Bender, S. Boettcher, and P. N. Meisinger, J. Math. Phys. **40**, 2201 (1999).
[8] A. Mostafazadeh, e-print arXiv:0810.5643; A. Mostafazadeh, J. Math Phys. **43**, 205 (2002); **43**, 2814 (2002); **43**, 3944 (2002); A. Mostafazadeh and A. Batal, J. Phys. A **37**, 11645 (2004); A. Mostafazadeh, Nucl. Phys. B **640**, 419 (2002).
[9] F. G. Scholtz, H. B. Geyer, and F. J. W. Hahne, Ann. Phys. **213**, 74 (1992).
[10] P. Dorey, C. Dunning, and R. Tateo, J. Phys. A **34**, 5679 (2001); **40**, R205 (2007).
[11] P. K. Ghosh, J. Phys. A **38**, 7313 (2005); P. K. Ghosh and K. S. Gupta, Phys. Lett. A **323**, 29 (2004); P. K. Ghosh, Eur. Phys. J. C **42**, 355 (2005).
[12] T. Deguchi and P. K. Ghosh, Phys. Rev. E **80**, 021107 (2009).
[13] Z. Ahmed and S. R. Jain, Phys. Rev. E **67**, 045106(R) (2003); J. Phys. A **36**, 3349 (2009); Mod. Phys. Lett. A **21**, 331 (2006); S. R. Jain and S. C. L. Srivastava, Phys. Rev. E **78**, 036213 (2008).
[14] K. Hepp and E. Lieb, Ann. Phys. (N.Y.) **76**, 360 (1973); Y. K. Wang and F. T. Hioe, Phys. Rev. A **7**, 831 (1973).
[15] M. Hillery and L. D. Mlodinow, Phys. Rev. A **31**, 797 (1985).
[16] C. Emary and T. Brandes, Phys. Rev. E **67**, 066203 (2003).
[17] N. Lambert, C. Emary, and T. Brandes, Phys. Rev. Lett. **92**, 073602 (2004).
[18] V. Buzek, M. Orszag, and M. Rosko, Phys. Rev. Lett. **94**, 163601 (2005); **96**, 089302 (2006).
[19] L. Amico and K. Hikami, Eur. Phys. J. B **43**, 387 (2005).
[20] S. Datta and B. Das, Appl. Phys. Lett. **56**, 665 (1990); M. Paternostro, G. Falci, M. Kim, and G. M. Palma, Phys. Rev. B **69**, 214502 (2004); F. Plastina and G. Falci, *ibid.* **67**, 224514 (2003).
[21] Figure 2(a) is not a perfect Poisson distribution. When $\alpha=\beta=\gamma=\delta=0$, the system is integrable, but the level statistics is non-Poissonian. All the nonzero level spacings are identical and the eigenvalues are highly degenerated. For small nonzero $\alpha\beta$, level statistics is influenced by the property.